

Characterization of Lipschitz spaces via commutators of the Hardy-Littlewood maximal function *

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Abstract Let M be the Hardy-Littlewood maximal function and b be a locally integrable function. Denote by M_b and $[b, M]$ the maximal commutator and the (nonlinear) commutator of M with b . In this paper, the author consider the boundedness of M_b and $[b, M]$ on Lebesgue spaces and Morrey spaces when b belongs to the Lipschitz space, by which some new characterizations of the Lipschitz spaces are given.

Keywords Hardy-Littlewood maximal function; commutator; Lipschitz space; Morrey space.

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1 Introduction and Results

Let T be the classical singular integral operator, the commutator $[b, T]$ generated by T and a suitable function b is given by

$$[b, T]f = bT(f) - T(bf). \quad (1.1)$$

A well known result due to Coifman, Rochberg and Weiss [6] (see also [13]) states that $b \in BMO(\mathbb{R}^n)$ if and only if the commutator $[b, T]$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. In 1978, Janson [13] gave some characterizations of the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ (see Definition 1.1 below) via commutator $[b, T]$ and proved that $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ ($0 < \beta < 1$) if and only if $[b, T]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ where $1 < p < n/\beta$ and $1/p - 1/q = \beta/n$ (see also Paluszyński [18]).

For a locally integrable function f , the Hardy-Littlewood maximal function M is given by

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

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the maximal commutator of M with a locally integrable function b is defined by

$$M_b(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |b(x) - b(y)| |f(y)| dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x .

The mapping properties of the maximal commutator M_b have been studied intensively by many authors. See [3], [9], [11], [12], [20], [21] and [26] for instance. The following result is proved by García-Cuerva et al. [9]. See also [20] and [21].

Theorem A ([9]) *Let b be a locally integrable function and $1 < p < \infty$. Then the maximal commutator M_b is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ if and only if $b \in BMO(\mathbb{R}^n)$.*

The first part of this paper is to study the boundedness of M_b when the symbol b belongs to Lipschitz space. Some characterizations of Lipschitz space via such commutator are given.

Definition 1.1 *Let $0 < \beta < 1$, we say a function b belongs to the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ if there exists a constant C such that for all $x, y \in \mathbb{R}^n$,*

$$|b(x) - b(y)| \leq C|x - y|^\beta.$$

The smallest such constant C is called the $\dot{\Lambda}_\beta$ norm of b and is denoted by $\|b\|_{\dot{\Lambda}_\beta}$.

Our first result can be stated as follows.

Theorem 1.1 *Let b be a locally integrable function and $0 < \beta < 1$, then the following statements are equivalent:*

- (1) $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.
- (2) M_b is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for all p, q with $1 < p < n/\beta$ and $1/q = 1/p - \beta/n$.
- (3) M_b is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for some p, q with $1 < p < n/\beta$ and $1/q = 1/p - \beta/n$.
- (4) M_b satisfies the weak-type $(1, n/(n - \beta))$ estimates, namely, there exists a positive constant C such that for all $\lambda > 0$,

$$|\{x \in \mathbb{R}^n : M_b(f)(x) > \lambda\}| \leq C(\lambda^{-1} \|f\|_{L^1(\mathbb{R}^n)})^{n/(n-\beta)}. \quad (1.2)$$

- (5) M_b is bounded from $L^{n/\beta}(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$.

Morrey spaces were originally introduced by Morrey in [17] to study the local behavior of solutions to second order elliptic partial differential equations. Many classical operators of harmonic analysis were studied in Morrey type spaces during the last decades. We refer the readers to Adams [2] and references therein.

Definition 1.2 Let $1 \leq p < \infty$ and $0 \leq \lambda \leq n$. The classical Morrey space is defined by

$$L^{p,\lambda}(\mathbb{R}^n) = \{f \in L^p_{\text{loc}}(\mathbb{R}^n) : \|f\|_{L^{p,\lambda}} < \infty\},$$

where

$$\|f\|_{L^{p,\lambda}} := \sup_Q \left(\frac{1}{|Q|^{\lambda/n}} \int_Q |f(x)|^p dx \right)^{1/p}.$$

It is well known that if $1 \leq p < \infty$ then $L^{p,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $L^{p,n}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$.

Theorem 1.2 Let b be a locally integrable function and $0 < \beta < 1$. Suppose that $1 < p < n/\beta$, $0 < \lambda < n - \beta p$ and $1/q = 1/p - \beta/(n - \lambda)$. Then $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ if and only if M_b is bounded from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$.

Theorem 1.3 Let b be a locally integrable function and $0 < \beta < 1$. Suppose that $1 < p < n/\beta$, $0 < \lambda < n - \beta p$, $1/q = 1/p - \beta/n$ and $\lambda/p = \mu/q$. Then $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ if and only if M_b is bounded from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\mu}(\mathbb{R}^n)$.

On the other hand, similar to (1.1), we can define the (nonlinear) commutator of the Hardy-Littlewood maximal function M with a locally integrable function b by

$$[b, M](f)(x) = b(x)M(f)(x) - M(bf)(x).$$

Using real interpolation techniques, Milman and Schonbek [16] established a commutator result. As an application, they obtained the L^p -boundedness of $[b, M]$ when $b \in BMO(\mathbb{R}^n)$ and $b \geq 0$. This operator can be used in studying the product of a function in H^1 and a function in BMO (see [5] for instance). In 2000, Bastero, Milman and Ruiz [4] studied the necessary and sufficient conditions for the boundedness of $[b, M]$ on L^p spaces when $1 < p < \infty$. Zhang and Wu obtained similar results for the fractional maximal function in [25] and extended the mentioned results to variable exponent Lebesgue spaces in [26] and [27]. Recently, Agcayazi et al. [3] gave the end-point estimates for the commutator $[b, M]$. Zhang [24] extended these results to the multilinear setting.

We would like to remark that operators M_b and $[b, M]$ essentially differ from each other. For example, M_b is positive and sublinear, but $[b, M]$ is neither positive nor sublinear.

The second part of this paper is to study the mapping properties of the (nonlinear) commutator $[b, M]$ when b belongs to some Lipschitz space. To state our results, we recall the definition of the maximal operator with respect to a cube. For a fixed cube Q_0 , the Hardy-Littlewood maximal function with respect to Q_0 of a function f is given by

$$M_{Q_0}(f)(x) = \sup_{Q_0 \supseteq Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

where the supremum is taken over all the cubes Q with $Q \subseteq Q_0$ and $Q \ni x$.

Theorem 1.4 *Let b be a locally integrable function and $0 < \beta < 1$. Suppose that $1 < p < n/\beta$ and $1/q = 1/p - \beta/n$. Then the following statements are equivalent:*

- (1) $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b \geq 0$.
- (2) $[b, M]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.
- (3) There exists a constant $C > 0$ such that

$$\sup_Q \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |b(x) - M_Q(b)(x)|^q dx \right)^{1/q} \leq C. \quad (1.3)$$

Theorem 1.5 *Let $b \geq 0$ be a locally integrable function, $0 < \beta < 1$ and $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$. Then there is a positive constant C such that for all $\lambda > 0$,*

$$|\{x \in \mathbb{R}^n : |[b, M](f)(x)| > \lambda\}| \leq C(\lambda^{-1} \|f\|_{L^1(\mathbb{R}^n)})^{n/(n-\beta)}.$$

Theorem 1.6 *Let b be a locally integrable function and $0 < \beta < 1$. Suppose that $1 < p < n/\beta$, $0 < \lambda < n - \beta p$ and $1/q = 1/p - \beta/(n - \lambda)$. Then the following statements are equivalent:*

- (1) $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b \geq 0$.
- (2) $[b, M]$ is bounded from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$.

Theorem 1.7 *Let b be a locally integrable function and $0 < \beta < 1$. Suppose that $1 < p < n/\beta$, $0 < \lambda < n - \beta p$, $1/q = 1/p - \beta/n$ and $\lambda/p = \mu/q$. Then the following statements are equivalent:*

- (1) $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b \geq 0$.
- (2) $[b, M]$ is bounded from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\mu}(\mathbb{R}^n)$.

This paper is organized as follows. In the next section, we recall some basic definitions and known results. In Section 3, we will prove Theorems 1.1—1.3. Section 4 is devoted to proving Theorems 1.4—1.7.

2 Preliminaries and Lemmas

For a measurable set E , we denote by $|E|$ the Lebesgue measure and by χ_E the characteristic function of E . For $p \in [1, \infty]$, we denote by p' the conjugate index of p , namely, $p' = p/(p - 1)$. For a locally integrable function f and a cube Q , we denote by $f_Q = (f)_Q = \frac{1}{|Q|} \int_Q f(x) dx$.

To prove the theorems, we need some known results. It is known that the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ coincides with some Morrey-Companato space (see [14] for example) and can be characterized by mean oscillation as the following lemma, which is due to DeVore and Sharpley [8] and Janson, Taibleson and Weiss [14] (see also Paluszyński [18]).

Lemma 2.1 *Let $0 < \beta < 1$ and $1 \leq q < \infty$. Define*

$$\dot{\Lambda}_{\beta,q}(\mathbb{R}^n) := \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : \|f\|_{\dot{\Lambda}_{\beta,q}} = \sup_Q \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^q dx \right)^{1/q} < \infty \right\}.$$

Then, for all $0 < \beta < 1$ and $1 \leq q < \infty$, $\dot{\Lambda}_\beta(\mathbb{R}^n) = \dot{\Lambda}_{\beta,q}(\mathbb{R}^n)$ with equivalent norms.

Let $0 < \alpha < n$ and f be a locally integrable function, the fractional maximal function of f is given by

$$\mathfrak{M}_\alpha(f)(x) = \sup_Q \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)| dy$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x .

The following strong and weak-type boundedness of \mathfrak{M}_α are well-known, see [10] and [7].

Lemma 2.2 *Let $0 < \alpha < n$, $1 \leq p \leq n/\alpha$ and $1/q = 1/p - \alpha/n$.*

(1) If $1 < p < n/\alpha$ then there exists a positive constant $C(n, \alpha, p)$ such that

$$\|\mathfrak{M}_\alpha(f)\|_{L^q(\mathbb{R}^n)} \leq C(n, \alpha, p) \|f\|_{L^p(\mathbb{R}^n)}.$$

(2) If $p = n/\alpha$ then there exists a positive constant $C(n, \alpha)$ such that

$$\|\mathfrak{M}_\alpha(f)\|_{L^\infty(\mathbb{R}^n)} \leq C(n, \alpha) \|f\|_{L^{n/\alpha}(\mathbb{R}^n)}.$$

(3) If $p = 1$ then there exists a positive constant $C(n, \alpha)$ such that for all $\lambda > 0$

$$|\{x \in \mathbb{R}^n : \mathfrak{M}_\alpha(f)(x) > \lambda\}| \leq C(n, \alpha) (\lambda^{-1} \|f\|_{L^1(\mathbb{R}^n)})^{n/(n-\alpha)}.$$

Spanne (see [19]) and Adams [1] studied the boundedness of the fractional integral I_α in classical Morrey spaces. We note that the fractional maximal function enjoys the same boundedness as that of the fractional integral since the pointwise inequality $\mathfrak{M}_\alpha(f)(x) \leq I_\alpha(|f|)(x)$. These results can be summarized as follows (see also [22]):

Lemma 2.3 *Let $0 < \alpha < n$, $1 < p < n/\alpha$ and $0 < \lambda < n - \alpha p$.*

(1) If $1/q = 1/p - \alpha/(n - \lambda)$ then there is a constant $C > 0$ such that

$$\|\mathfrak{M}_\alpha(f)\|_{L^{q,\lambda}(\mathbb{R}^n)} \leq C \|f\|_{L^{p,\lambda}(\mathbb{R}^n)} \quad \text{for every } f \in L^{p,\lambda}(\mathbb{R}^n).$$

(2) If $1/q = 1/p - \alpha/n$ and $\lambda/p = \mu/q$. Then there is a constant $C > 0$ such that

$$\|\mathfrak{M}_\alpha(f)\|_{L^{q,\mu}(\mathbb{R}^n)} \leq C \|f\|_{L^{p,\lambda}(\mathbb{R}^n)} \quad \text{for every } f \in L^{p,\lambda}(\mathbb{R}^n).$$

Lemma 2.4 ([15]) *Let $1 \leq p < \infty$ and $0 < \lambda < n$, then there is a constant $C > 0$ that depends only on n such that*

$$\|\chi_Q\|_{L^{p,\lambda}(\mathbb{R}^n)} \leq C |Q|^{\frac{n-\lambda}{np}}.$$

3 Proof of Theorems 1.1—1.3

Proof of Theorem 1.1 If $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, then

$$\begin{aligned}
 M_b(f)(x) &= \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |b(x) - b(y)| |f(y)| dy \\
 &\leq C \|b\|_{\dot{\Lambda}_\beta} \sup_{Q \ni x} \frac{1}{|Q|^{1-\beta/n}} \int_Q |f(y)| dy \\
 &= C \|b\|_{\dot{\Lambda}_\beta} \mathfrak{M}_\beta(f)(x).
 \end{aligned} \tag{3.1}$$

Obviously, (2), (3), (4) and (5) follow from Lemma 2.2, Lemma 2.3 and (3.1).

(3) \implies (1): Assume M_b is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for some p, q with $1 < p < n/\beta$ and $1/q = 1/p - \beta/n$. For any cube $Q \subset \mathbb{R}^n$, by Hölder's inequality and noting that $1/p + 1/q' = 1 + \beta/n$, one gets

$$\begin{aligned}
 \frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - b_Q| dx &\leq \frac{1}{|Q|^{1+\beta/n}} \int_Q \left(\frac{1}{|Q|} \int_Q |b(x) - b(y)| dy \right) dx \\
 &= \frac{1}{|Q|^{1+\beta/n}} \int_Q \left(\frac{1}{|Q|} \int_Q |b(x) - b(y)| \chi_Q(y) dy \right) dx \\
 &\leq \frac{1}{|Q|^{1+\beta/n}} \int_Q M_b(\chi_Q)(x) dx \\
 &\leq \frac{1}{|Q|^{1+\beta/n}} \left(\int_Q [M_b(\chi_Q)(x)]^q dx \right)^{1/q} \left(\int_Q \chi_Q(x) dx \right)^{1/q'} \\
 &\leq \frac{C}{|Q|^{1+\beta/n}} \|M_b\|_{L^p \rightarrow L^q} \|\chi_Q\|_{L^p(\mathbb{R}^n)} \|\chi_Q\|_{L^{q'}(\mathbb{R}^n)} \\
 &\leq C \|M_b\|_{L^p \rightarrow L^q}.
 \end{aligned}$$

This together with Lemma 2.1 gives $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.

(4) \implies (1): We assume (1.2) is true and will verify $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$. For any fixed cube $Q_0 \subset \mathbb{R}^n$, since for any $x \in Q_0$,

$$|b(x) - b_{Q_0}| \leq \frac{1}{|Q_0|} \int_{Q_0} |b(x) - b(y)| dy,$$

then, for all $x \in Q_0$,

$$\begin{aligned}
 M_b(\chi_{Q_0})(x) &= \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |b(x) - b(y)| \chi_{Q_0}(y) dy \\
 &\geq \frac{1}{|Q_0|} \int_{Q_0} |b(x) - b(y)| \chi_{Q_0}(y) dy \\
 &= \frac{1}{|Q_0|} \int_{Q_0} |b(x) - b(y)| dy \\
 &\geq |b(x) - b_{Q_0}|.
 \end{aligned}$$

This together with (1.2) gives

$$\begin{aligned} |\{x \in Q_0 : |b(x) - b_{Q_0}| > \lambda\}| &\leq |\{x \in Q_0 : M_b(\chi_{Q_0})(x) > \lambda\}| \\ &\leq C(\lambda^{-1} \|\chi_{Q_0}\|_{L^1(\mathbb{R}^n)})^{n/(n-\beta)} \\ &= C(\lambda^{-1} |Q_0|)^{n/(n-\beta)}. \end{aligned}$$

Let $t > 0$ be a constant to be determined later, then

$$\begin{aligned} \int_{Q_0} |b(x) - b_{Q_0}| dx &= \int_0^\infty |\{x \in Q_0 : |b(x) - b_{Q_0}| > \lambda\}| d\lambda \\ &= \int_0^t |\{x \in Q_0 : |b(x) - b_{Q_0}| > \lambda\}| d\lambda \\ &\quad + \int_t^\infty |\{x \in Q_0 : |b(x) - b_{Q_0}| > \lambda\}| d\lambda \\ &\leq t|Q_0| + C \int_t^\infty (\lambda^{-1} |Q_0|)^{n/(n-\beta)} d\lambda \\ &\leq t|Q_0| + C|Q_0|^{n/(n-\beta)} \int_t^\infty \lambda^{-n/(n-\beta)} d\lambda \\ &\leq C(n, \beta)(t|Q_0| + |Q_0|^{n/(n-\beta)} t^{1-n/(n-\beta)}). \end{aligned}$$

Set $t = |Q_0|^{\beta/n}$ in the above estimate, we have

$$\int_{Q_0} |b(x) - b_{Q_0}| dx \leq C|Q_0|^{1+\beta/n}.$$

It follows from Lemma 2.1 that $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ since Q_0 is an arbitrary cube in \mathbb{R}^n .

(5) \implies (1): If M_b is bounded from $L^{n/\beta}(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$, then for any cube $Q \subset \mathbb{R}^n$,

$$\begin{aligned} \frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - b_Q| dx &\leq \frac{1}{|Q|^{1+\beta/n}} \int_Q \left(\frac{1}{|Q|} \int_Q |b(x) - b(y)| \chi_Q(y) dy \right) dx \\ &\leq \frac{1}{|Q|^{1+\beta/n}} \int_Q M_b(\chi_Q)(x) dx \\ &\leq \frac{1}{|Q|^{\beta/n}} \|M_b(\chi_Q)\|_{L^\infty(\mathbb{R}^n)} \\ &\leq \frac{C}{|Q|^{\beta/n}} \|M_b\|_{L^{n/\beta} \rightarrow L^\infty} \|\chi_Q\|_{L^{n/\beta}(\mathbb{R}^n)} \\ &\leq C \|M_b\|_{L^{n/\beta} \rightarrow L^\infty}. \end{aligned}$$

This together with Lemma 2.1 gives $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.

The proof of Theorem 1.1 is completed since (2) \implies (1) follows from (3) \implies (1). \square

Proof of Theorem 1.2 Assume $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$. By (3.1) and Lemma 2.3 (1), we have

$$\|M_b(f)\|_{L^{q,\lambda}} \leq \|b\|_{\dot{\Lambda}_\beta} \|\mathfrak{M}_\beta(f)\|_{L^{q,\lambda}} \leq C \|b\|_{\dot{\Lambda}_\beta} \|f\|_{L^{p,\lambda}}.$$

Conversely, if M_b is bounded from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$, then for any cube $Q \subset \mathbb{R}^n$,

$$\begin{aligned}
\frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |b(x) - b_Q|^q dx \right)^{1/q} &\leq \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q \left[\frac{1}{|Q|} \int_Q |b(x) - b(y)| \chi_Q(y) dy \right]^q dx \right)^{1/q} \\
&\leq \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q [M_b(\chi_Q)(x)]^q dx \right)^{1/q} \\
&= \frac{1}{|Q|^{\beta/n}} \left(\frac{|Q|^{\lambda/n}}{|Q|} \right)^{1/q} \left(\frac{1}{|Q|^{\lambda/n}} \int_Q [M_b(\chi_Q)(x)]^q dx \right)^{1/q} \\
&\leq |Q|^{-\beta/n-1/q+\lambda/(nq)} \|M_b(\chi_Q)\|_{L^{q,\lambda}(\mathbb{R}^n)} \\
&\leq C |Q|^{-\beta/n-1/q+\lambda/(nq)} \|M_b\|_{L^{p,\lambda} \rightarrow L^{q,\lambda}} \|\chi_Q\|_{L^{p,\lambda}(\mathbb{R}^n)} \\
&\leq C \|M_b\|_{L^{p,\lambda} \rightarrow L^{q,\lambda}},
\end{aligned}$$

where in the last step we have used $1/q = 1/p - \beta/(n - \lambda)$ and Lemma 2.4.

It follows from Lemma 2.1 that $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$. This completes the proof. \square

Proof of Theorem 1.3 By a similar proof to the one of Theorem 1.2, we can obtain Theorem 1.3. \square

4 Proof of Theorems 1.4—1.7

Proof of Theorem 1.4 (1) \implies (2): For any fixed $x \in \mathbb{R}^n$ such that $M(f)(x) < \infty$, since $b \geq 0$ then

$$\begin{aligned}
|[b, M](f)(x)| &= |b(x)M(f)(x) - M(bf)(x)| \\
&= \left| \sup_{Q \ni x} \frac{1}{|Q|} \int_Q b(x)|f(y)|dy - \sup_{Q \ni x} \frac{1}{|Q|} \int_Q b(y)|f(y)|dy \right| \\
&\leq \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |b(x) - b(y)||f(y)|dy \\
&= M_b(f)(x).
\end{aligned} \tag{4.1}$$

It follows from Theorem 1.1 that $[b, M]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ since $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.

(2) \implies (3): For any fixed cube $Q \subset \mathbb{R}^n$ and all $x \in Q$, we have (see the proof of Proposition 4.1 in [4], see also (2.4) in [25])

$$M(\chi_Q)(x) = \chi_Q(x) \quad \text{and} \quad M(b\chi_Q)(x) = M_Q(b)(x).$$

Then,

$$\begin{aligned}
 & \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |b(x) - M_Q(b)(x)|^q dx \right)^{1/q} \\
 &= \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |b(x)M(\chi_Q)(x) - M_Q(b\chi_Q)(x)|^q dx \right)^{1/q} \\
 &= \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |[b, M](\chi_Q)(x)|^q dx \right)^{1/q} \\
 &\leq \frac{1}{|Q|^{\beta/n+1/q}} \| [b, M](\chi_Q) \|_{L^q(\mathbb{R}^n)} \\
 &\leq \frac{C}{|Q|^{\beta/n+1/q}} \| \chi_Q \|_{L^p(\mathbb{R}^n)} \\
 &\leq C,
 \end{aligned} \tag{4.2}$$

which implies (3) since the cube $Q \subset \mathbb{R}^n$ is arbitrary.

(3) \implies (1): To prove $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, by Lemma 2.1, it suffices to verify that there is a constant $C > 0$ such that for all cubes Q ,

$$\frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - b_Q| dx \leq C. \tag{4.3}$$

For any fixed cube Q , let $E = \{x \in Q : b(x) \leq b_Q\}$ and $F = \{x \in Q : b(x) > b_Q\}$. The following equality is trivially true (see [4] page 3331):

$$\int_E |b(x) - b_Q| dx = \int_F |b(x) - b_Q| dx.$$

Since for any $x \in E$ we have $b(x) \leq b_Q \leq M_Q(b)(x)$, then for any $x \in E$,

$$|b(x) - b_Q| \leq |b(x) - M_Q(b)(x)|.$$

Thus,

$$\begin{aligned}
 \frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - b_Q| dx &= \frac{1}{|Q|^{1+\beta/n}} \int_{E \cup F} |b(x) - b_Q| dx \\
 &= \frac{2}{|Q|^{1+\beta/n}} \int_E |b(x) - b_Q| dx \\
 &\leq \frac{2}{|Q|^{1+\beta/n}} \int_E |b(x) - M_Q(b)(x)| dx \\
 &\leq \frac{2}{|Q|^{1+\beta/n}} \int_Q |b(x) - M_Q(b)(x)| dx.
 \end{aligned} \tag{4.4}$$

On the other hand, it follows from Hölder's inequality and (1.3) that

$$\begin{aligned}
 \frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - M_Q(b)(x)| dx &\leq \frac{1}{|Q|^{1+\beta/n}} \left(\int_Q |b(x) - M_Q(b)(x)|^q dx \right)^{1/q} |Q|^{1/q'} \\
 &\leq \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |b(x) - M_Q(b)(x)|^q dx \right)^{1/q} \\
 &\leq C.
 \end{aligned}$$

this together with (4.4) gives (4.3) and so we achieve $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.

In order to prove $b \geq 0$, it suffices to show $b^- = 0$, where $b^- = -\min\{b, 0\}$. Let $b^+ = |b| - b^-$, then $b = b^+ - b^-$. For any fixed cube Q , observe that

$$0 \leq b^+(x) \leq |b(x)| \leq M_Q(b)(x)$$

for $x \in Q$ and therefore we have that, for $x \in Q$,

$$0 \leq b^-(x) \leq M_Q(b)(x) - b^+(x) + b^-(x) = M_Q(b)(x) - b(x).$$

Then, it follows from (1.3) that, for any cube Q ,

$$\begin{aligned} \frac{1}{|Q|} \int_Q b^-(x) dx &\leq \frac{1}{|Q|} \int_Q |M_Q(b)(x) - b(x)| \\ &\leq \left(\frac{1}{|Q|} \int_Q |b(x) - M_Q(b)(x)|^q dx \right)^{1/q} \\ &= |Q|^{\beta/n} \left\{ \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |b(x) - M_Q(b)(x)|^q dx \right)^{1/q} \right\} \\ &\leq C|Q|^{\beta/n}. \end{aligned}$$

Thus, $b^- = 0$ follows from Lebesgue's differentiation theorem.

The proof of Theorem 1.4 is completed. \square

Proof of Theorem 1.5 Obviously, Theorem 1.5 follows from (4.1) and Theorem 1.1. \square

Proof of Theorem 1.6 (1) \implies (2): Assume $b \geq 0$ and $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, then by (4.1) and Theorem 1.2 we see that $[b, M]$ is bounded from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$.

(2) \implies (1): Assume that $[b, M]$ is bounded from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$. Similarly to (4.2), we have, for any cube $Q \subset \mathbb{R}^n$,

$$\begin{aligned} &\frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |b(x) - M_Q(b)(x)|^q dx \right)^{1/q} \\ &= \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |[b, M](\chi_Q)(x)|^q dx \right)^{1/q} \\ &\leq \frac{|Q|^{\lambda/(nq)}}{|Q|^{\beta/n+1/q}} \| [b, M](\chi_Q) \|_{L^{q,\lambda}(\mathbb{R}^n)} \\ &\leq \frac{C|Q|^{\lambda/(nq)}}{|Q|^{\beta/n+1/q}} \| \chi_Q \|_{L^{p,\lambda}(\mathbb{R}^n)} \\ &\leq C, \end{aligned}$$

where in the last step we have used $1/q = 1/p - \beta/(n - \lambda)$ and Lemma 2.4.

This shows by Theorem 1.4 that $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b \geq 0$. \square

Proof of Theorem 1.7 By the same way of the proof of Theorem 1.6, Theorem 1.7 can be proven. We omit the details. \square

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